

Sampling of analog signals

1. Real and ideal sampling

Consider the principle module of a sampling circuit, as shown in figure 1, consisting of a switch which lets the analogue signal $x_a(t)$ pass from input to output for very short intervals, τ , around times $t = nT$, by closing the switch K .

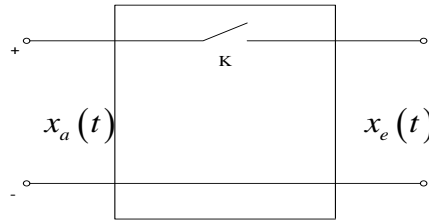


Figure 1 Ideal sampling circuit

Figure 2 shows the time variations of the analogue input signal and the sampled output signal.

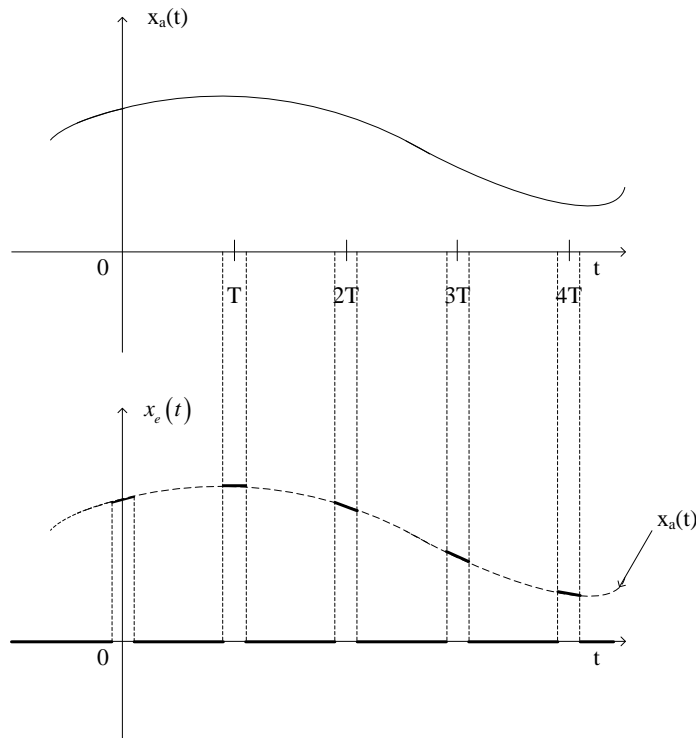


Figure 2 . $x_a(t)$ -analogue signal at sample circuit input, $x_e(t)$ -sampled signal (sample circuit output result)

The sampled signal can be useful for multiple use of the transmission path. In the breaks between its pulses, the telecommunications path becomes clear, and samples of other analog signals can be transmitted (time multiplexing). In order for this process to be as efficient as possible, it would be desirable that the duration $T - \tau$ during which samples of the signal under consideration are not transmitted be as long as possible, i.e. τ as short and T as high as possible.

The reduction of pulse durations in the sampled signal is limited, among other things, by increasing, beyond acceptable limits, the frequency bandwidth occupied by this signal (according to the property of expansion or contraction in time or frequency of the Fourier transform of analog signals).

On the other hand, if the sampling period becomes too long, different analogue signals may occur, which means that the sampled signal will no longer represent only a certain analogue signal, which will therefore not be able to be unequivocally restored at reception from its samples.

2. Sampling theorem

The discrete signal $x[n]$, resulting from analogue signal sampling $x_a(t)$, is defined:

$$x[n] = x_e(nT) = x_a(nT).$$

Denoting $X_a(\omega)$ Fourier transform of analog signal $x_a(nT)$, one can write the Fourier transform of the discrete signal:

$$X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\omega + k \frac{2\pi}{T}\right).$$

In this process, two distinct situations can arise, depending on whether in the discrete signal spectrum the areas where the different frequency-shifted replicas of the nonzero-value analog signal spectrum overlap or not, as illustrated in the following figure.

In the analogue signal spectrum, it was considered to have an effective frequency band limited to a maximum value f_M (or $\omega_M = 2\pi f_M$).

The two situations mentioned, $2\Omega_M \leq \Omega = \frac{2\pi}{T}$ and $2\Omega_M > \Omega = \frac{2\pi}{T}$, are illustrated in Figure 3.

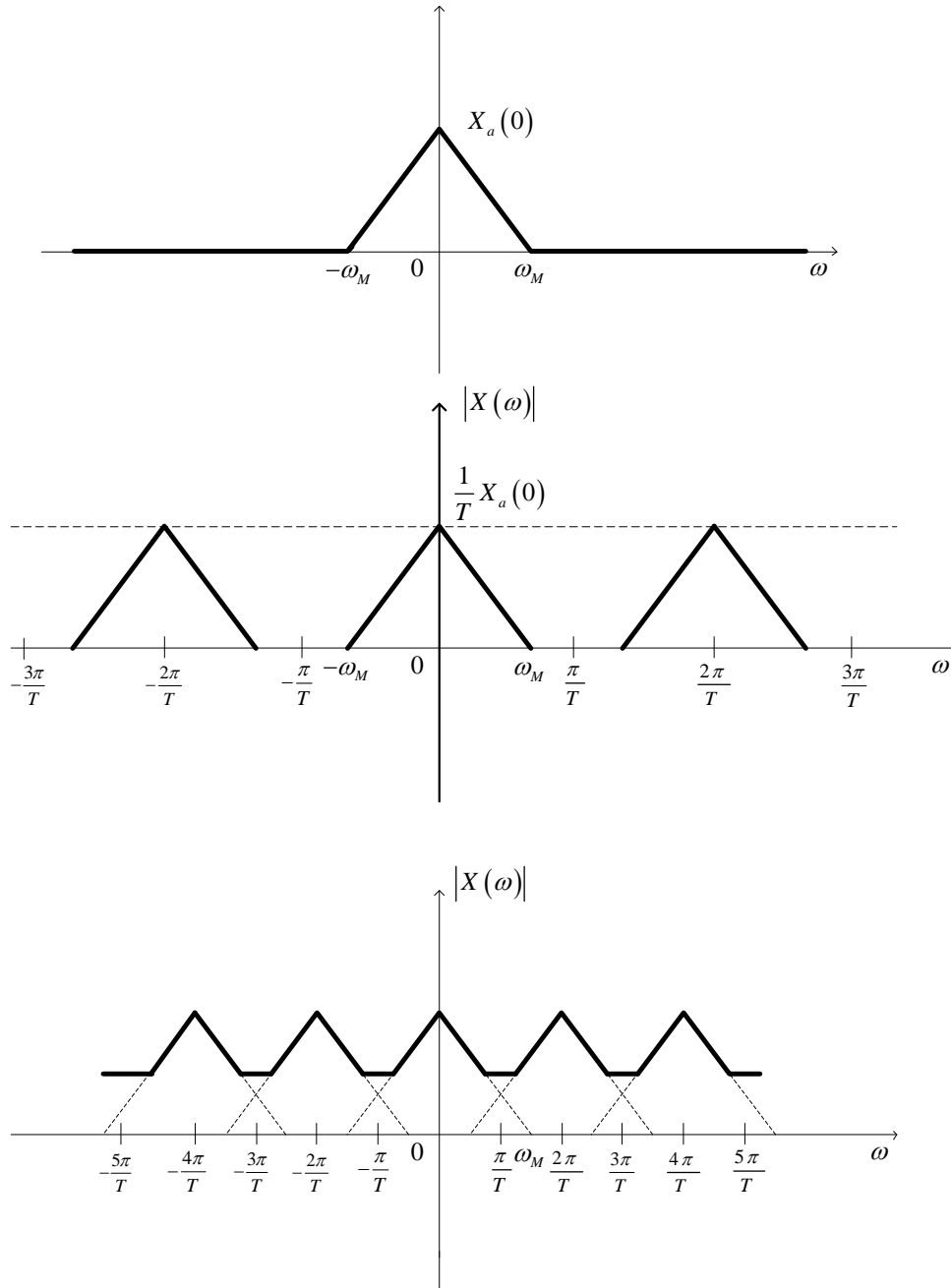


Figure 3 a. $X_a(\omega)$ Fourier transform of analog signal, b. Fourier transform modulus of discrete signal for

$$\omega_M \leq \frac{\pi}{T}, \text{ c. Fourier transform modulus of discrete signal for } \omega_M > \frac{\pi}{T}$$

It is found that, if $\omega_M \leq \frac{\pi}{T}$, from the spectrum of the discrete signal can be separated a portion in which it is proportional to the spectrum of the analog signal. Consequently, using a low-

pass filter with cutting pulsation $\frac{\pi}{T}$, one can univocally recover the analog signal from its samples, because:

$$X(\omega) = \frac{1}{T} X_a(\omega) \text{ for } \omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T} \right]$$

In the latter case, $\omega_M > \frac{\pi}{T}$, this procedure is no longer applicable due to the partial overlap between the spectrum of the analogue signal and its replicas shifted on the frequency axis.

The fact that in the first case the signal $x_a(t)$ can be obtained, by low-pass filtering, from the discrete signal $x[n]$, assumes that $x_a(t)$ may be reconstituted from its samples $x_a(nT)$ through an integration formula.

$$x_a(t) = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} TX(\omega) e^{j\omega t} d\omega.$$

Knowing that the Fourier transform in the case of discrete signals (TFSD), with ideal samples, taken at times $t_n = nT$, of analog signals is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-jnT\omega} = \sum_{n=-\infty}^{\infty} x[n] e^{-jnT\omega},$$

When the sum converges for $\forall \omega \in \mathbb{R}$, it follows:

$$x_a(t) = \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} T \sum_{n=-\infty}^{\infty} x(nT) e^{-jnT\omega} e^{j\omega t} d\omega$$

Substituting $x(nT) = x_a(nT)$ and changing the order of integration and summation operations, ensue:

$$\begin{aligned} x_a(t) &= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x_a(nT) \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{j(t-nT)\omega} d\omega = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x_a(nT) \frac{e^{j(t-nT)\omega} \Big|_{-\frac{\pi}{T}}^{\frac{\pi}{T}}}{j(t-nT)} = \\ &= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x_a(nT) \frac{e^{j(t-nT)\frac{\pi}{T}} - e^{-j(t-nT)\frac{\pi}{T}}}{j(t-nT)} = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x_a(nT) \frac{2 \sin \left[(t-nT) \frac{\pi}{T} \right]}{t-nT} \end{aligned}$$

A non-periodic analog signal, limited bandwidth, at a maximum frequency f_M , is fully defined by samples extracted at discrete times $nT, n \in \mathbb{Z}$, if the sampling period T is less than $\frac{1}{2f_M} \left(T \leq \frac{1}{2f_M} \right)$ or $F = \frac{1}{T}$ is at least twice the maximum frequency in the analogue signal spectrum ($F \geq 2f_M$). This statement constitutes the sampling theorem (Shannon's theorem), and the value $2f_M$ is referred to as the Nyquist sampling cut-off frequency.

Example 1

a. Using Matlab generate the discrete sequence $x[n]$ obtained from the signal $x(t) = \cos(2\pi F_1 t)$ of frequency $F_1 = 3\text{kHz}$ and duration $t_{max} = 100$ ms, by sampling it with frequency $F_s = 8\text{kHz}$.

b. Plot the amplitude spectrum of TFSD in unnormed frequencies $F \in [0,8)$ [kHz] using the `fft()` function. Why do we have components at 3kHz and 5kHz?

c. Redo subpoint b using the `fftshift()` function for $F \in [-4,4)$ [kHz]. Why do we have components at -3kHz and 3kHz in this case?

```

clc; clear; close all
F1 = 3000;
Fs = 8000; % sampling frequency
Ts = 1/Fs; % sampling period
Tmax = 0.1; % signal duration
t = 0: Ts: Tmax-Ts; % time vector
signal = cos(2*pi*F1*t);

nr_esantioane = length(signal);

N = 64; %length of Fourier transform
spectrum = fft(signal, N); % Fourier transform of sampled signal

F = 0:Fs/N :Fs-Fs/N;
figure(1),stem(F,abs(spectrum),grid
xlabel('Frequency (Hz)');
ylabel('Amplitude');
title('Signal spectrum');

F = -Fs/2: Fs/N :Fs/2-Fs/N;
figure(2),stem(F,fftshift(abs(spectrum)),grid
xlabel('Frequency (Hz)');
ylabel('Amplitude');
title('Signal spectrum');

```

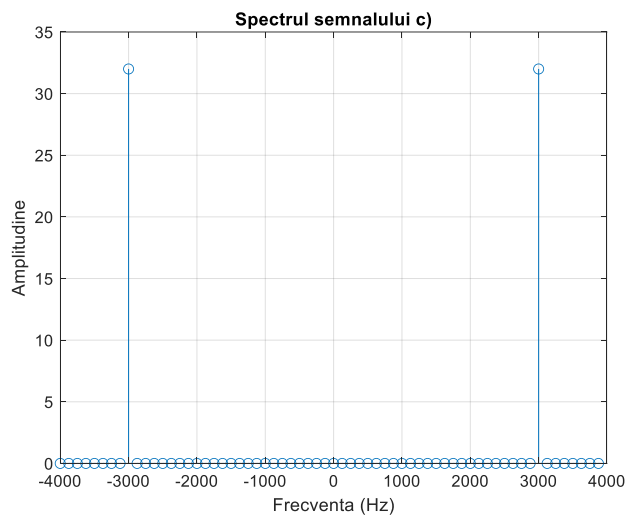
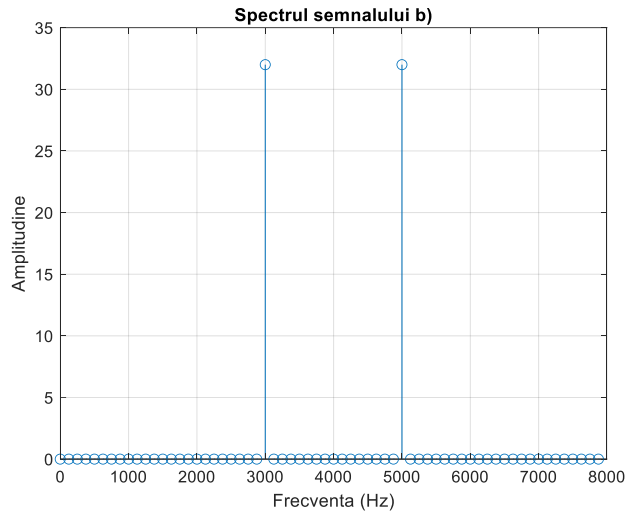


Figure 4 signal spectrum using `fft()`, signal spectrum using `fftshift()`

Example 2

Using Matlab represent in the signal time domain $x(t)$, the sampled signal with the frequency 1000 Hz $x(t)$ and the signal $x_a(t)$ restored from its samples

$$x(t) = \begin{cases} \cos(2\pi \cdot 100 \cdot t), & 0 < t < 100 \text{ ms} \\ 0, & \text{otherwise} \end{cases}$$

```
clear all
close all
clc
```

```
Fs=1000; % sampling frequency
```

```

Ts=1/Fs; % sampling period
Tmax=0.1; % signal duration
Nmax=Tmax*Fs;

F=100;
td=0:Ts:Tmax;
sd=cos(2*pi*F*td);

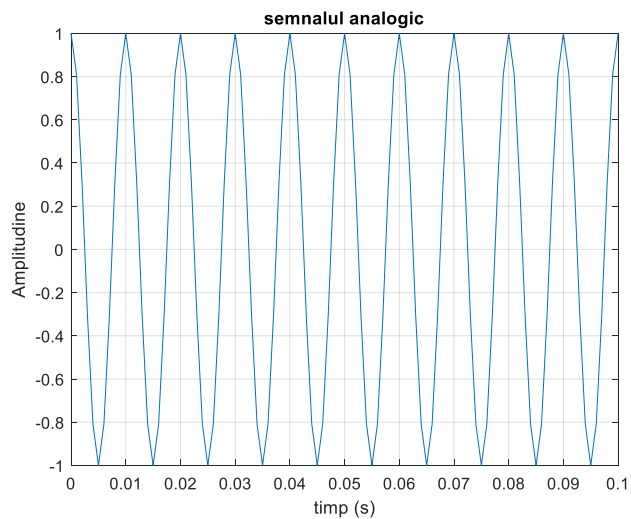
figure(1),plot(td,sd),grid
xlabel('time(s)');
ylabel('Amplitude');
title('analogue signal');

figure(2),stem(td,sd),grid
xlabel('time(s)');
ylabel('Amplitude');
title('sample signal');

ta=0:0.000001:Tmax;
for n=0:Nmax-1
    H(:,n+1)=sd(n+1)*2*sin(pi*Fs*(ta-n*Ts))./(ta-n*Ts);
End

sa=Ts/(2*pi)*sum(H,2);
figure(3),plot(ta,sa),grid
xlabel('time(s)');
ylabel('Amplitude');
title('signal restored');

```



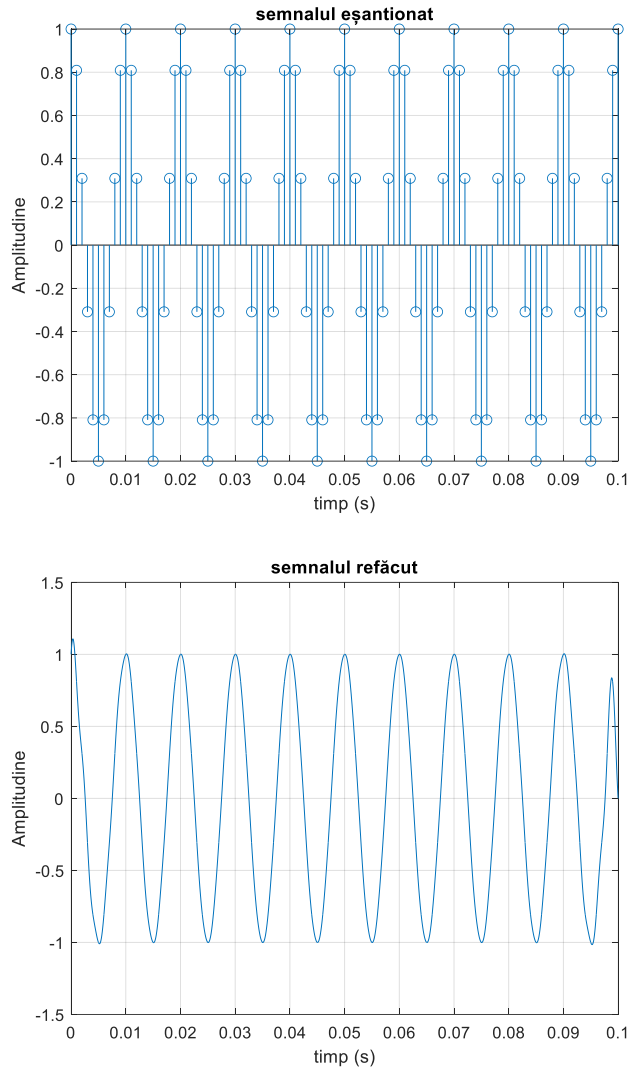


Figure 5 analogue signal, sampled signal, signal restored from its samples

Exercise 1

- Using Matlab generate the discrete sequence $x[n]$ obtained from the signal $x(t) = \cos(\pi \cdot F_1 \cdot t) + 2 \cos(2\pi \cdot F_1 \cdot t)$, of frequency $F_1 = 3\text{kHz}$ and duration $t_{max} = 100\text{ms}$, by sampling it with $F_s = 8\text{kHz}$
- Plot the amplitude spectrum of TFSD in unnormalized frequencies $F \in [0,8)$ [kHz] using the `fft()` function.
- Redo subpoint b using the `fftshift()` function for $F \in [-4,4)$ [kHz].

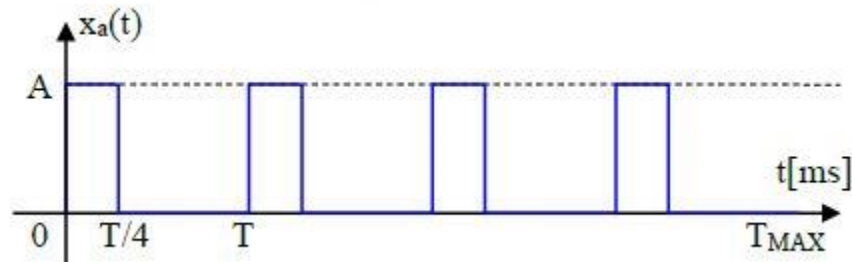
Exercise 2

Repeat example 2 for the $x(t)$ 800Hz sampled signal:

$$x(t) = \begin{cases} 2 * \cos(2\pi \cdot 50 \cdot t) + 2 * \cos(2\pi \cdot 25 \cdot t), & 0 < t < 100 \text{ ms} \\ 0, & \text{otherwise} \end{cases}$$

Exercise 3

- a. Generate the discrete signal $x[n]$ obtained by sampling $x_a(t)$ with sampling frequency $F_s = 30 \text{ kHz}$, $T = 1/F_0$, $F_0 = 3.5 \text{ kHz}$, acquisition duration $T_{\text{max}} = 10\text{ms}$, $A = 3$.



- b. Plot with the stem function (abscissa according to n) the discrete signal $x[n]$.
- c. Plot with the plot function (abscissa as a function of time in milliseconds) the analog $x(t)$ signal restored by analog digital conversion from discrete signal.
- d. Determine: the total number of samples L for $x[n]$ and the number of samples N in a period T . How many periods k are included in the acquisition duration T_{max} .
- e. Calculate the TFSD of the $x[n]$ signal over a number of points equal to the length $N_{\text{fft}} = 1024$
- a. Plot the amplitude and phase spectrum in non-standard frequencies F in the range $[0, F_s)$ [kHz]
- i. Determine on the graph the fundamental harmonic corresponding to the fundamental frequency F_0 as well as the frequencies F_n corresponding to the harmonics. At what frequencies do harmonic components occur?
- ii. Determine on the graph the amplitudes of spectral components corresponding to the continuous component, fundamental F_0 and harmonics. What relationship is there between the amplitude A of the signal and the amplitudes measured on the graph?
- f. Perform a function that calculates the fill factor of the rectangular signal

Exercise 4

Generate the discrete $x[n]$ signal obtained by sampling an analogue audio.

```
[x, Fs] = audioread('f2bjrop1.0.wav');
```

- a. Listen to the signal using the function: `sound(x,Fs)`
- b. Calculate the TFSD of the $x[n]$ signal over a number of points equal to the length $N_{\text{fft}} = 1024$ and plot the amplitude spectrum in non-standard frequencies F in the range $[0, F_s)$ [kHz]

- c. Generate a sine wave signal w of amplitude 0.005 of frequency F_{sin} large enough not to overlap with the spectral components of the audio signal ($F_{\text{min}} \ll F_{\text{sin}} < F_s/2$), of the same dimensions as the $x[n]$ signal
- d. Add this w signal on top of the original signal, x_{new}
- e. Calculate the TFSD of the signal $x_{\text{new}}[n]$ over a number of points equal to the length N_{fft} 1024 and plot the amplitude spectrum in non-standard frequencies F in the range $[0, F_s)$ [kHz]
 - a. You can identify the spectrum position of the added signal
- f. Listen to the new signal. Can you hear the added signal?

Bibliography

Teodor Petreacu, Signals and Systems, POLITEHNICA PRESS București, 2019

Matlab Online Documentation, <https://matlabacademy.mathworks.com/artifacts/quick-reference.html?course=mlsg&language=en&release=R2023b>